Follow-up from fractals session

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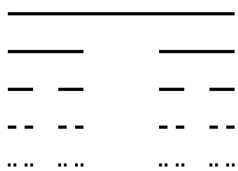
These questions – in the style of investigations – are designed to make you think further about some of the ideas we explored in the session. They are not necessarily easy! You might find it useful to revisit the recording of the session.

Questions with a (*) in front of them are somewhat harder.

1. The Cantor set is formed by starting with a line segment, then removing the middle third of it, then repeatedly removing the middle third of the remaining segments; the first four stages of this process are shown here:

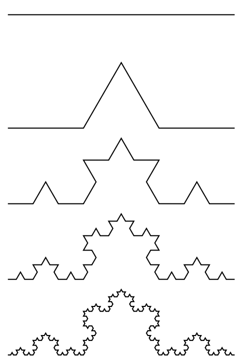
   What is the dimension of the resulting fractal?

   What happens if instead of removing the middle third you remove the middle quarter each time, or a different (but fixed) fraction?

   Can you find a variant of this that has dimension exactly $\frac{1}{2}$?

   (*) What happens if instead of removing the middle third you remove a third of each segment each time, but starting from half way along (so leaving two pieces, one is $\frac{1}{2}$ of the original line and the other is $\frac{1}{6}$)?

2. The von Koch curve is constructed as follows. Start with a straight line segment. Replace the middle third of it with the other two sides of an equilateral triangle. Now repeat this process on every segment; the first four steps are shown here:

What is the dimension of the resulting fractal?

If you start with an equilateral triangle instead of a single line segment, the resulting shape is known as the von Koch snowflake. You might enjoy drawing it!

(*) What happens if instead of removing the middle third of each segment, you remove the middle quarter each time, or a different (but fixed) fraction less than $\frac{1}{3}$, and replace the removed part with the other two sides of an equilateral triangle?

3. We stated the following theorem: If a shape is made of similar (non-overlapping) copies of itself, with scale factors s_1, s_2, …, s_k, then the dimension d of the shape is given by the equation $s_1^d + s_2^d + \cdots + s_k^d = 1$.

   Check that if we think of a line segment as n copies of itself, each scaled by a factor of $\frac{1}{n}$, then this theorem states that the dimension of the line segment is 1 (as we would hope). Likewise, show that if a square is divided into copies of itself scaled by a factor of $\frac{1}{n}$, then this theorem states that the dimension of the square is 2, and similarly for a cube.

We may or may not have had time to explore the Mandelbrot set during the session. If you want to learn (more) about complex numbers and the Mandelbrot set, Charlie Gilderdale and Claire Metcalfe from NRICH ran a webinar for the 2021 Cambridge Science Festival, which you can watch here: `https://youtu.be/oM7MKRznj44`. This also has a link to a GeoGebra applet investigating the Mandelbrot set: `https://www.geogebra.org/calculator/fqpabfsu`.

In addition to the question below, you might also find Question 4 from the STEP Support Programme Foundation Assignment 18 interesting: `https://maths.org/step/assignments/step-support-assignment-18`.

4. We explored the iteration given by $f(x) = x^2 - 1$. When we start with $x = 2$, the iterates get bigger and bigger: $x = 2$, then $x = f(2) = 3$, then $x = f(3) = 8$, then $x = f(8) = 63$ and so on, and similarly when we start with $x = -2$. If we start with a value of x greater than 2, the same sort of thing happens.

   Can you find a starting value of x for which the iterates never get as large as 2? What is the largest value of x for which this is the case? (This point marks the largest point in the Julia set for -1.)

You might be interested in exploring some other fractals. Two relatively simple yet fascinating ones are the dragon curve and the Fibonacci word fractal; you can find out lots about them on the internet.